JOURNAL OF AEROSPACE COMPUTING, INFORMATION, AND COMMUNICATION
Vol. 6, April 2009

New Algorithm for Probabilistic Robustness Analysis in Parameter Space

Gianfranco Morani* , Federico Corraro† , and Antonio Vitale‡
Italian Aerospace Research Centre, Capua, (CE), 81043, Italy

DOI: 10.2514/1.39338

In this paper a new algorithm for robustness analysis of uncertain parametric systems is proposed. The algorithm adopts a probabilistic approach to find a multidimensional region in the uncertainty parameter space where a system satisfies a given property. In particular it finds a (suboptimal) maximum volume hyper-rectangle in which the given system's property is satisfied with a pre-assigned confidence. The algorithm has been applied for the robustness analysis of the Italian Aerospace Research Centre's unmanned space vehicle demonstrator's maneuverability. The use of the algorithm during the design phase of the project lowered the effort spent in aerodynamic wind tunnel testing on specific coefficients that did not require the reduction of the uncertainty ranges. Moreover the algorithm has been used for estimating the flying test bed's initial state displacement compatible with a safe mission execution to support online launch decisions. Effectiveness of the proposed method in terms of computational efficiency and reliability has been demonstrated by comparing the results with a deterministic method that finds the actual region in the uncertainty space where the system properties are verified. The proposed algorithm allows the computational burden of robustness analysis to be drastically reduced, particularly when the number of uncertain parameters is greater than three.

Nomenclature

C_D	drag coefficient
C_L	lift coefficient
C_m	pitching moment coefficient
$f(q)$	multivariate probability density function
l	dimension of uncertain parameter space
M	Mach number
N	number of random samples
p_0, q_0, r_0	initial angular velocities, deg/s
q	vector of uncertain parameters
Q	uncertain parameter space
Q_{good}	subset of uncertainty space Q in which a given property is satisfied
Q_{fin}^{opt}	probabilistic estimate of the maximum volume hyper-rectangle included in Q_{good}

Received 26 June 2008; revision received 19 December 2008; accepted for publication 19 December 2008.

* Researcher, Flight Systems Department, g.morani@cira.it.
† Senior Researcher, Flight Systems Department, f.corraro@cira.it.
‡ Researcher, Flight Systems Department, a.vitale@cira.it.

Q_{true}^{opt}	maximum volume hyper-rectangle probabilistically included in Q_{good}
Q_0^{opt}	maximum volume hyper-rectangle probabilistically included in Q_{good} and having the same shape of Q
Q_ε	subset of uncertainty space Q in which a given property is satisfied with probability greater than 1-ε
$r_{max}(\varepsilon)$	probabilistic robustness margin
Re	Reynolds number
$u(q)$	generic function of uncertain parameters q
u_{max}	worst-case value of the function $u(q)$
u_N	estimate of worst-case value of the function $u(q)$
α	angle of attack, deg
γ	performance level
ΔC_D	uncertainty on drag coefficient
ΔC_L	uncertainty on lift coefficient
ΔC_m	uncertainty on pitching moment coefficient
Δx_{cg}	uncertainties on the Center of Gravity x-position, m
Δz_{cg}	uncertainties on the Center of Gravity z-position, m
δ	confidence of probabilistic estimate
δ_e	elevons deflection, deg
ε	accuracy of probabilistic estimate
θ_0	initial pitch angle, deg
ϕ_0	initial roll angle, deg
ψ_0	initial heading angle, deg

I. Introduction

ITALIAN Aerospace Research Centre (CIRA) is conducting an aerospace national research program named USV (unmanned space vehicle), whose main objective is designing and manufacturing unmanned flying test beds (FTB), conceived as multimission flying laboratories, to test and verify innovative materials, aerodynamic methods, advanced guidance, navigation and control functionalities, and critical operational aspects peculiar of the future reusable launch vehicle (RLV) and aerospace plane. A problem of paramount importance to be dealt with during the USV project development phases was the analysis of the uncertainties characterizing FTB1 system (the first experimental FTB of USV project). In fact, a common problem that arises during the design phase of a system, and when verifying the control performance and stability robustness, is to assess system compliance with a set of predefined requirements in the presence of some process parametric uncertainties ranging in given subsets. This problem is particularly important and challenging when the final system performs safety critical functions, as in aerospace vehicles [1] and nuclear or chemical plants application areas. In many cases, especially during early stages of a design process, it can be useful to estimate the uncertainty ranges that still satisfy a requirement. In this way critical design parameters (which require the reduction of the uncertainty ranges) can be identified [2] and the detailed design activities and the uncertainty refinement process can be optimally planned to obtain, at the end of the implementation process, uncertainty ranges that still allow requirements compliance. The above stated problem is peculiar to USV project because the parametric uncertainties, mainly the aerodynamics ones, are significantly large. USV missions are, in fact, designed to fly in supersonic and transonic regimes in which the aerodynamic parameter values and behavior are difficult to predict through computational fluid dynamics (CFD) and wind tunnel tests. Therefore in the framework of such a project two kinds of problem arise. First of all, the large aerodynamic uncertainties call for early analyses aimed at estimating the uncertainty ranges that still satisfy the requirements such as maneuverability to plan activities (such as changes of the vehicle configuration or further wind tunnel tests) concerning possible uncertainty refinement. Moreover, because guidance, navigation and control (GNC) design has the aim of guaranteeing robustness to all uncertainties affecting the system and the mission (aerodynamic, mass and inertia, environmental, initial state, etc.), an effective analysis approach is required for assessing final robustness and for identifying mission parameters (i.e. initial states displacement, winds, etc.), which could be of major concern for the mission success.

When dealing with robustness analysis of an uncertain system, the following key problem can be defined. Given an uncertain system, a region defined in the uncertain parameter space and some properties or requirements to be verified, we may estimate the actual region in the uncertain parameter space where the system robustly satisfies the requirements (robustness analysis problem). The robustness analysis problem can be considered of paramount importance for improving the system robustness to uncertainties because it gives an indication of critical areas in the uncertainty space where the system design should be improved.

In the past years, several research activities have been performed in the field of robustness analysis with respect to parametric uncertain systems, especially concerning robust stability and performance of linear feedback systems (see [1] and [3] for a comprehensive discussion). The concept of multivariable stability margin, which represents the largest perturbation (according to a given norm) still guaranteeing system's stability, and structured singular value [4,5] are introduced to deal with structured perturbations. In this setting all the uncertainties affecting the system are arranged into canonical block diagonal feedback perturbations (M–Δ configuration). Regarding robustness analysis with parametric uncertainty, in the last decades several methods have been developed (see [6] and [7]). These methods address the problem of robustness analysis starting from the transfer function of an uncertain plant, nevertheless most of their results apply only in the case that the coefficients of the closed-loop characteristic polynomial can be expressed as affine maps of the uncertain parameters, which is in general a restrictive hypothesis. In [8] an algorithm is proposed that allows us to estimate a multidimensional uncertainty region in the parameters space in which a given property (expressed through the inequality $u(q) \leqslant \gamma$) is satisfied in the case that $u(q)$ is a multi-affine mapping of the uncertain parameters q. Therefore the common approaches to the robustness analysis only work with linear time invariant (LTI) systems and with limited uncertainty structure. This dramatically limits application of the above mentioned approaches to practical cases. The general case of nonlinear mapping is addressed in [9] and [10]. Although more efficient than the commonly used gridding techniques [1], this method suffers from the limitation of having a complexity which is exponentially growing with the number of uncertain parameters.

Computational complexity is in fact an issue of major concern for all the methods presented above. Recent research activities on this subject have shown that various robust control problems are NP-hard [11].

In view of these considerations, recently a new formulation of the robustness analysis problem has grown in importance [12,13]. Instead of dealing with estimation of the actual robustness region, it may be more useful to pursue a robustness analysis problem solution which guarantees that a certain property holds for "most" of the uncertainty combinations, that is, a given property can be violated by a set of uncertainties having small probability measure. Even if this change in problem formulation is not sufficient to make the robustness analysis problem simpler (the probability of satisfying a given property may be very hard to compute because it requires the evaluation of a multidimensional integral), it allows us to estimate the probability that a given property is satisfied by means of randomized algorithms (RA) (see [14–18]). RA are algorithms that make some random choices during their execution and they accomplish the robustness analysis, performing random extractions of uncertainty samples and using them to achieve a probabilistic estimation of satisfaction of a given system property.

Although these algorithms overcome some limitations of classical robustness analysis methods, mainly the computational complexity and the inapplicability to every system and every uncertainty structure, they mostly aim to find the probabilistic counterpart of the multivariable robustness margin above mentioned previously. Nevertheless this robustness indicator has the limitation of underestimating the system's robustness because it evaluates how much the uncertainty space can be expanded by the same factor in all directions still ensuring that a given property is satisfied. Clearly removing this limitation may provide a less conservative estimation of the system's robustness capabilities. This problem is discussed in recent researches [19] aimed at providing a framework to allow for different bounds for each uncertainty component. The concept of volumetric singular value (VSV) is introduced so that the multivariable stability margin is defined as the maximum achievable volume in which stability is preserved. The authors argue that structured singular value is a suboptimal solution compared with VSV because it can be viewed as the solution to the problem of volume maximization with the geometric constraint that all the uncertainty bounds must be scaled by the same factor.

Starting from the concept of VSV, in this paper a new algorithm is proposed that allows the problem of robustness analysis to be solved. It relies on a probabilistic approach to inherit all the advantages previously described (reduced complexity, applicability to all kind of systems, etc.), but unlike other approaches it looks for the whole region in the uncertain parameter space satisfying the system's requirements instead of computing only the robustness margin.

In more detail, the algorithm aims to find the maximum volume hyper-rectangle in which a given system property is satisfied with pre-assigned accuracy and confidence. The algorithm can be used for robustness analysis problems applied to every system, even nonlinear time variant and for every requirement or property that can be evaluated numerically or analytically. Furthermore, regarding the computational burden, it will be shown that it is polynomial in the dimension of uncertainty space (i.e. it increases with polynomial law as the dimension of the uncertainty space increases).

Therefore the algorithm is capable of overcoming the main limitations of the existing approaches to the robustness analysis problem, namely:

- high degree of conservatism introduced with the concept of multivariable robustness margin discussed above
- inapplicability to nonLTI systems
- prohibitive computational complexity.

Nevertheless, as will be discussed in the following sections, the identified hyper-rectangle is not guaranteed to be the absolute maximum volume hyper-rectangle but it is in general a suboptimal solution because it depends on the shape of the initial uncertainty space. Furthermore it is worth to note that the problem of finding the maximum volume hyper-rectangle provides a suboptimal solution itself, because in general this hyper-rectangle is included in the actual region (that is not usually a hyper-rectangle) where an assigned system property is satisfied. The computation of this region as well as the computation of the maximum volume hyper-rectangle can be performed by means of deterministic techniques, but they require a high computational burden. On the contrary, as it will be explained in the following sections, the computation of a suboptimal volume hyper-rectangle combined with the use of the probabilistic approach allows a significant reduction of computational complexity if compared with other deterministic techniques. Moreover it is worth to remark that the algorithm always provides a hyper-rectangle with volume greater than or equal to the one obtained by scaling all the uncertainties by the same factor (as done for the computation of the multivariable robustness margin), thus giving more information about the robustness capabilities of the system under analysis.

The proposed algorithm has been successfully used in the framework of CIRA USV project, in particular, to address the following two problems:

- maneuverability analysis for USV-FTB1 vehicle during early design phase
- robustness analysis of FTB1 flight control laws (FCL) to initial state displacements during the GNC verification phase.

Maneuverability analysis made it possible to identify the critical flight envelope points where aerodynamic database should be refined, because maneuverability requirement was not satisfied on the whole range of uncertainty. Following the results provided by the proposed algorithm, further wind tunnel tests and CFD analyses have been focused on the refinement of selected aerodynamic uncertainties in few critical flight envelope points, thus drastically reducing the wind tunnel testing costs and time.

Robustness analysis of FTB1 FCL allowed easy identification of the subset of initial state uncertainty region still guaranteeing the achievement of mission objectives, and these results have been used during the first FTB1 mission to decide online whether the vehicle initial state was suitable for safely performing the flight.

The proposed algorithm's effectiveness has been demonstrated by comparing the results with the deterministic method [8–10] that finds the actual region in the uncertainty space where the system properties are verified. It will be shown that even if the proposed algorithm provides a suboptimal solution, it guarantees a significant amount of computational complexity reduction, which becomes more noticeable when the number of uncertain parameters grows.

In Sec. II robustness analysis in a probabilistic setting is introduced and a detailed description of the proposed algorithm is provided together with its advantages and limitations. In Sec. III applications for the above stated robustness analysis problems are presented and finally in Sec. IV some brief concluding remarks are included.

II. Robustness Analysis in a Probabilistic Framework

Consider a system with uncertain parameters $q \hat{=} (q_1, q_2, \ldots, q_l) \in Q \subset \Re^l$ and given bounds $|q_i| \leqslant I_i$ for $i = 1, 2, \ldots, l$. The random vector q has a multivariate probability density function (pdf) $f(q)$ and its components q_i are assumed to be independent and can be viewed as deviations from the nominal value, so they are centered at $q_i = 0$. Moreover we can assume that $I_i = 1, \forall i$ via an appropriate scaling without loss of generality. With these assumptions, the uncertainty region Q is the hypercube in the l-dimensional parameter space $Q = I_1 \times I_2 \times \cdots \times I_l$ with $I_i = [-1, +1]$, $i = 1, 2, \ldots, l$.

Generally speaking, the problem of robustness analysis in the uncertain parameter space can be formulated as follows. Given the uncertainty space Q, an uncertain system, a function of uncertain parameters $u(q)$ defined in Q and a generic system property expressed through the inequality $u(q) \leqslant \gamma$ (where γ is an assigned threshold), determine the robustness set $Q_{\text{good}} = \{q \in Q : u(q) \leqslant \gamma\}$. It is worth noting that the function $u(q)$ is a generic function of uncertain parameters (hereinafter called performance function), so it can be related to the system's performances, stability and to every property which can be expressed through the inequality $u(q) \leqslant \gamma$.

In the previous section we described the various approaches which tried to solve the above stated problem or to find regions of pre-assigned shape included in Q_{good}. In this paper, we formulate the above problem in a probabilistic way, that is, we find a set Q_ε where property $u(q) \leqslant \gamma$ is probabilistically satisfied, i.e. $\Pr\{u(q) \leqslant \gamma\} \geqslant 1 - \varepsilon$, $q \in Q_\varepsilon$. Therefore we propose a probabilistic approach to find the maximum volume hyper-rectangle included in Q_{good}, that is

$$\begin{gathered} \max_{\bar{R} \in \Re^l} \prod_{i=1}^{l} R_i \\ \text{subject to} \\ \bar{R} Q \subseteq Q_{\text{good}} \subseteq Q \end{gathered} \tag{1}$$

where $\bar{R} \stackrel{\text{def}}{=} [R_1, R_2, \ldots, R_l]$ and $\bar{R} Q \stackrel{\text{def}}{=} R_1 I_1 \times R_2 I_2 \times \cdots \times R_l I_l$. It is worth noting that in this setting the relation $\bar{R} Q \subseteq Q_{\text{good}}$ is intended to be from a probabilistic point of view, i.e. $\bar{R} Q \subseteq Q_{\text{good}} \Leftrightarrow \Pr\{u(q) \leqslant \gamma\} \geqslant 1 - \varepsilon$, $q \in \bar{R} Q$.

The above stated problem is of great importance for the robustness analysis because finding the (maximum volume) subset of the uncertainty space in which a system property is satisfied gives a direct measure of a system's robustness. For instance, in the case of a closed-loop system, we may evaluate the maximum capabilities of the control algorithms to fulfill performance or stability by reacting to the system's uncertainties.

A. System Property Verification

To solve the problem stated in the previous section, we need a way to evaluate if the system robustly verifies the system's property when the uncertainties q are allowed to vary in a generic set $Q' \subseteq Q \subset \Re^l$ and are described by the pdf $f(q)$. An efficient method to accomplish the above analysis is to estimate the worst-case value of the performance function $u(q)$

$$u_{\max} = \max_{q \in Q'} u(q) \tag{2}$$

The estimation can be performed by means of a Monte Carlo sampling as follows

$$u_N = \max_{i=1,2,\ldots,N} u(q^{(i)}) \tag{3}$$

where $q^{(i)}$ are i.i.d. samples drawn in Q' according to $f(q)$. It can be shown [16,20] that if

$$N \geqslant \frac{\ln(1/\delta)}{\ln(1/1-\varepsilon)} \tag{4}$$

then

$$\Pr\{\Pr\{u(q) > u_N\} \leqslant \varepsilon\} \geqslant 1 - \delta \tag{5}$$

Therefore if $u_N \leqslant \gamma$, with a probability greater than $1 - \delta$ the system property is probabilistically satisfied in Q', that is, $\Pr\{u(q) \leqslant \gamma\} \geqslant 1 - \varepsilon$, $q \in Q'$.

It is worth noting that the above results are dependent on the pdf $f(q)$ used to describe the random vector $\boldsymbol{q}$. Nevertheless $f(q)$ may not be known a priori, so a question arises about which is the best choice for the pdf (see [21] for further discussions). In this regard, we observe that the uniform pdf is widely used for the robustness analysis problem owing to its worst-case properties (see [22,23]). Nevertheless when a uniform sampling in a given region is performed, the generated random samples are more likely to be concentrated on the surface of the region, if the number of uncertain parameters is high [24]. Therefore a uniform pdf might not be the best choice when the points of the uncertainty space having the value of the performance function greater than γ (i.e. the points which do not satisfy the system's property) are located far from the surface of the uncertainty region, because in this case the worst-case value of Eq. (2) may be underestimated. Therefore a different pdf could be used, depending on the specific problem to be solved.

In any case when we lack some a priori knowledge about the distribution of the non-compliant points in the uncertainty space Q' where the sampling is performed, the use of the uniform pdf allows us to state that, with probability greater than $1-\delta$, the volume of the "non compliant" set $Q_\gamma = \{q \in Q' : u(q) > \gamma\}$ is at most ε times the volume of Q' (see [23]), that is

$$\Pr\{\mathrm{vol}(Q_\gamma) \leqslant \varepsilon \mathrm{vol}(Q')\} \geqslant 1-\delta \tag{6}$$

even if the probability that the desired property is not satisfied, i.e. $\Pr\{u(q) > \gamma\}, q \in Q'$ may be greater than ε depending on the *actual* probability distribution of the random vector q.

B. Algorithm Description

Once the uncertainty distribution to be used is defined and a way to check if a prespecified property is verified in a given uncertainty region, it is possible to describe the steps towards the search of the maximum volume hyper-rectangle included in the unknown region Q_{good}. The proposed algorithm is based on the adaptive random search technique detailed below.

In the first step we find a solution for the following simplified robustness analysis problem. Let us define the subspace $Q_r = rQ$ with $r \in [0, 1]$. The aim of the first step is finding the *probabilistic robustness margin* defined as follows

$$r_{\max}(\varepsilon) \hat{=} sup\{r : \Pr\{u(q) \leqslant \gamma\} \geqslant 1-\varepsilon, q \in Q_r\} \tag{7}$$

This can be viewed as the robustness analysis problem (1) in the special case when the set Q_{good} has the same shape of Q. This formulation is useful because it allows the l-dimensional problem of Eq. (1) to be reduced into a one-parameter search problem so to reduce the computational charge.

The problem can be solved by means of a worst-case estimation (see Eq. (3)) as well. In fact to find the largest r such that $Q_r = rQ$ is (probabilistically) included in Q_{good}, we can simply find the largest r such that $u_N \leqslant \gamma$. In fact if N is chosen according to Eq. (4), then with a confidence greater than $1-\delta$, $u(q)$ will be less than u_N (i.e. the property will be satisfied) with probability greater than $1-\varepsilon$ for $q \in rQ$ (see inequality (5)).

Therefore starting from $r = 1$, the algorithm explores a subset rQ of the uncertainty space Q by generating Nrandom samples in this set and evaluating if the performance requirement $u(q^{(i)}) \leqslant \gamma$ is satisfied; whenever a sample vector violating the property is found, the radius r is reduced such that the new radius is the largest radius, which does not include the noncompliant dispersed point. Therefore this simplified algorithm (hereafter called Algorithm 1) makes it possible for us to find the maximum set included in Q_{good} having the same shape of the uncertainty hypercube Q and satisfying the system property, that is

$$Q_0^{\mathrm{opt}} = r_{\max} I_1 \times r_{\max} I_2 \times \cdots \times r_{\max} I_l \subseteq Q_{\mathrm{good}} \subset \Re^l \tag{8}$$

Finally Algorithm 1 can be also used to scale only m bounds ($m < l$) of the uncertainty space Q, that is, to scale the set $\hat{Q} \subset \Re^m$ and letting the remaining $l-m$ components to vary in a fixed range $\tilde{Q} \subset \Re^{l-m}$. As it will be shown, this is useful when computing a hyper-rectangle included in Q_{good}, which is not constrained to have the same shape of the initial uncertainty space Q. A meta-code of the Algorithm 1 is given below.

Algorithm 1.
Need to specify the number of samples N, $f(q)$, $u(q)$, γ, the sets $\hat{Q}$(to be scaled) and $\tilde{Q}$ (fixed set).

1. Initialize $r_{\max} = 1$.
2. Let $k = 1$.
3. Generate a vector sample $q^{(k)}$ according to $f(q)$ in $\tilde{Q} \times r_{\max}\hat{Q} \subset \Re^l$.

4. **If** system's property is not satisfied, that is, if $u\left(q^{(k)}\right) > \gamma$, **then** let $r_{\max} = \max_{i=1,2,\ldots,l} \left|q_i^{(k)}\right|$ and go to step 2
   else if $k < N$ **then** let $k = k + 1$and go to step 3.
5. End, return the variable $r_{\max}$.

It is worth noting that if $\hat{Q} = Q$ and $\tilde{Q} = []$ all the components of the uncertainty hypercube are scaled in the same way and Q_0^{opt} of Eq. (8) is obtained.

Obviously finding a solution to the robustness analysis problem which accounts for different bounds for each component q_i could provide more information about the robustness of the system under analysis. As already mentioned, to address the more general problem of finding a hyper-rectangle included in the set Q_{good} which is not constrained to have the same shape of Q, further steps are needed. To keep the problem complexity low, we may compute a larger hyper-rectangle starting from Q_0^{opt} by iteratively repeating the Algorithm 1 on uncertainty space of reduced dimension, thus obtaining the following Algorithm 2.

Algorithm 2.
Need to specify the starting scale factor R, the number of samples N, $f(q)$, $u(q)$, γ, the set $\bar{Q} = I_{k_1} \times I_{k_2} \times \cdots \times I_{k_h} \subset \Re^h$ (to be scaled) and $Q' \subset \Re^{l-h}$ (fixed set).

1. **If** $h > 1$
   For $i = 1, 2, \ldots, h$
   Let $r_{k_i} = R$ and compute $r_{\max}$ using Algorithm 1 with $\hat{Q} = \bar{Q} - \{I_{k_i}\} \overset{\text{def}}{=} I_{k_1} \times \cdots \times I_{k_{i-1}} \times I_{k_{i+1}} \times \cdots \times I_{k_h} \subset \Re^{h-1}$, $\tilde{Q} = Q' \times r_{k_i} I_{k_i} \subset \Re^{l-(h-1)}$, N, $f(q)$, $u(q)$, γ.
   Let $r_i^{\max} = r_{\max}$
   end
   else return R
2. Take the index n such that $r_n^{\max} = \max_{i=1,2,\ldots,h} r_i^{\max}$.
3. End, return the index k_n and $r_n^{\max}$.

If $h = l$ Algorithm 2 is applied to the whole starting uncertainty hypercube, that is $\bar{Q} = Q$, $Q' = []$ with R given by Algorithm 1 with $\hat{Q} = Q$ and $\tilde{Q} = []$, that is $R = r_{\max}$. Therefore the ith bound of Q is set at $r_{\max}$ and Algorithm 1 is repeated on the remaining $l - 1$ bounds, that is, on the set $Q_i = I_1 \times \cdots \times I_{i-1} \times I_{i+1} \times \cdots \times I_l \subset \Re^{l-1}$. Thus for each component q_i a radius $r_i^{\max} \geqslant r_{\max}$is found (through Algorithm 1) such that the (larger) hyper-rectangle

$$Q_{r_i^{\max}} = r_i^{\max} I_1 \times \cdots \times r_i^{\max} I_{i-1} \times r_{\max} I_i \times r_i^{\max} I_{i+1} \times \cdots \times r_i^{\max} I_l \subset \Re^l \tag{9}$$

is (probabilistically) included in Q_{good}. Algorithm 1 is iteratively repeated for $i = 1, 2, \ldots, l$, and the uncertainty bound I_n allowing the largest expansion (in the remaining $l - 1$ dimension) $r_n^{\max} = \max_i r_i^{\max}$ is selected and finally fixed at $r_{\max}$, so the updated solution after the first iteration of Algorithm 2 will be

$$Q^{\text{opt}} = r_n^{\max} I_1 \times \cdots \times r_n^{\max} I_{n-1} \times r_{\max} I_n \times r_n^{\max} I_{n+1} \times \cdots \times r_n^{\max} I_l \subseteq Q_{\text{good}} \subset \Re^l \tag{10}$$

Algorithm 2 is now iteratively used to determine the bounds of the other $l - 1$ uncertainty components, that is $I_1, I_2, \ldots I_{n-1}, I_{n+1}, \ldots I_l$ and compute the final optimum hyper-rectangle. In fact, at each step the solution previously found is refined to achieve a larger hyper-rectangle. In detail, starting from Q^{opt} in Eq. (10), Algorithm 2 is repeated on all the bounds except the nth. Hence for each $j = 1, 2, \ldots, n-1, n+1, \ldots, l$ the jth bound of $\bar{Q} = Q - \{I_n\} \overset{\text{def}}{=} I_1 \times \cdots \times I_{n-1} \times I_{n+1} \times \cdots \times I_l \subset \Re^{l-1}$ is set at $r_n^{\max}$ and a new radius $r_j^{\max} \geqslant r_n^{\max} \geqslant r_{\max}$ is computed using Algorithm 1. The "best" radius $r_k^{\max} = \max_j r_j^{\max}$ is selected as done before. Therefore the updated solution after two iterations of Algorithm 2 will be

$$\begin{aligned} Q^{\text{opt}} = {} & r_k^{\max} I_1 \times \cdots \times r_k^{\max} I_{k-1} \times r_n^{\max} I_k \times r_k^{\max} I_{k+1} \times \cdots \times r_k^{\max} I_{n-1} \times r_{\max} I_n \times r_k^{\max} I_{n+1} \\ & \times \cdots \times r_k^{\max} I_l \subseteq Q_{\text{good}} \subset \Re^l \end{aligned} \tag{11}$$

In the subsequent iterations, Algorithm 2 is repeated on the set $\bar{Q} = Q - \{I_k \times I_n\} \overset{\text{def}}{=} I_1 \times \cdots \times I_{k-1} \times I_{k+1} \times \cdots \times I_{n-1} \times I_{n+1} \times \cdots \times I_l \subset \Re^{l-2}$ and so on until all the bounds of the uncertainty space are fixed. It is worth noting that at each iteration the updated solution is a suboptimum of problem (1). This solution is obtained solving a specific maximization problem and it has, by construction, a greater volume than the solution found in the previous step. In this way the final solution $Q_{\text{fin}}^{\text{opt}}$ will satisfy the following inequality

$$\text{vol}(Q_{\text{fin}}^{\text{opt}}) \geqslant \text{vol}(Q_0^{\text{opt}}) \tag{12}$$

so Q_0^{opt} is a lower bound of the suboptimal solution computed by the algorithm. In conclusion the whole procedure is described by the following meta-code:

Algorithm 3 (main).
Need to specify the starting uncertainty hypercube $Q \subset \Re^l$, the number of samples N, $f(q)$, $u(q)$, γ.

1. Let $h = l$, $\bar{Q} = Q$, $Q' = []$, $\tilde{Q} = []$.
2. Compute $r_{\max}$using Algorithm 1 with $\hat{Q} = Q$, $\tilde{Q}$, N, $f(q)$, $u(q)$, ?
3. Let $R = r_{\max}$.
4. **While** $h > 1$
5. Compute k_n and $r_n^{\max}$ using Algorithm 2 with $\bar{Q}$, Q', R, N, $f\ (q)$, $u\ (q)$, γ.
6. Let $R_{k_n} = R$, $R = r_n^{\max}$, $\bar{Q} = \bar{Q} - \{I_{k_n}\} \subset \Re^{h-1}$, $Q' = R_{k_n} I_{k_n} \times Q' \subset \Re^{l-(h-1)}$
7. Let $h = h - 1$
8. **end**
9. Return the set $Q_{\text{fin}}^{\text{opt}} = Q' \times R\bar{Q}$.

As mentioned previously, the relationship between $Q_{\text{fin}}^{\text{opt}}$ and Q_{good} is to be intended in a probabilistic sense, that is, the following inequality holds

$$\Pr\{\Pr\{u(q) \leqslant \gamma\} \geqslant 1 - \varepsilon, q \in Q_{\text{fin}}^{\text{opt}}\} \geqslant 1 - \delta \tag{13}$$

that is, with probability greater than $1 - \delta$ the set $Q_{\text{fin}}^{\text{opt}}$ will have a probability greater than $1 - \varepsilon$ of being compliant with the required system property.

C. Advantages and Limitations

The main advantage of the proposed algorithm is its reduced computational charge. To address the complexity issue, the following preliminary considerations shall be made:

- Algorithm 1 has a computational complexity, which does not depend on the number of uncertain parameters l.
- During the execution of the proposed algorithm, Algorithm 1 is executed a number of times $M = \sum_{i=1}^{l} i = l(l+1)/2$ which is polynomial in the dimension of uncertainty space.

In view of these considerations, we argue that the proposed algorithm has a complexity growing polynomial with the dimension of the uncertainty space. This result dramatically reduces the computational burden of the proposed algorithm compared with deterministic techniques. Furthermore it is worth noting that the algorithm can be slightly modified by treating the uncertainty space Q as a space of dimension $2l$, that is, $Q = I_1 \times \bar{I}_1 \times I_2 \times \bar{I}_2 \cdots \times I_l \times \bar{I}_l$ with $I_i = [-1, 0]$, $\bar{I}_i = [0, +1]$, $i = 1, 2, \ldots, l$. In this way we can better account for those sets Q_{good} that are not symmetric around the nominal value with a computational complexity still polynomial, because it is sufficient to replace l with $2l$ in the above formula. Finally another significant advantage concerns the algorithm applicability to any system and any property to be verified.

Regarding algorithm limitations, it shall be noted that the above described procedure leads to a suboptimal solution of problem (1). In fact, as previously explained, at each step the solution is obtained maximizing the volume of admissible uncertainties while retaining the shape of the initial uncertainty space.

For this reason the solution will be suboptimal because of the dependence of the solution on the shape of the initial uncertainty space. In any case we obtain a solution whose volume is always greater or equal than that of the lower bound Q_0^{opt} in Eq. (7) as explained before, that is

$$\text{vol}(Q_0^{\text{opt}}) \leqslant \text{vol}(Q_{\text{fin}}^{\text{opt}}) \leqslant \text{vol}(Q_{\text{true}}^{\text{opt}}) \tag{14}$$

where $Q_{\text{true}}^{\text{opt}}$ is the solution of the problem stated in Eq. (1). However as will be shown in the following sections, the suboptimality of the solution is counteracted by a significant saving of the computational effort.

III. Application to USV FTB 1 System Robustness Analysis

To fulfill the first part of CIRA USV program, three flight missions were planned and they will be performed by a couple of identical FTBs (FTB1 configuration) being designed and manufactured to support the USV program execution. In this section two examples of robustness analyses carried out during the development of USV program will be provided, in particular with reference to the first mission named DTFT (dropped transonic flight test) 1.

DTFT1 has been successfully carried out on February 2007 in Sardinia, Italy. The main aims of this mission were the investigation of the vehicle aerodynamic behavior and in-flight validation of flight control system in the transonic flight regime. In DTFT1 the FTB1 vehicle was dropped from an altitude of about 20 km and performed a controlled flight to reach a Mach value of 1.05.

A. Maneuverability Analysis for USV FTB 1 Vehicle

The proposed algorithm was used to perform an open-loop assessment of the USV-FTB1 vehicle robust maneuverability. The maneuverability property accounts for the possibility to change the flight trajectory and vehicle attitude during all phases of the flight. This property can be investigated by verifying that the aerodynamic rotational moment of the vehicle, over all the foreseen flight conditions, changes its sign while moving the relevant control surfaces within their allowable ranges. Specifically, the position of the elevons (that are deputed for controlling the longitudinal flight) for which the rotational moment changes its sign shall be sufficiently far from the end of travel limits (maneuverability margin). In our case, the elevons' nominal travel range was (± 20 deg) and the minimum allowable maneuverability margin was fixed to 25% of the elevons' nominal range. In this way, in the worst case, the control system could always use the 25% of the elevons' travel range, to change the current attitude.

In the early design phase of the USV project, a preliminary aerodynamic database was available, which provided the global longitudinal aerodynamic coefficients for different values of angle of attack α, elevons deflection δ_e, Mach number M and Reynolds number Re. All the aerodynamic coefficients were considered as independent variables and defined by a most likely value (nominal value) and an uncertainty range. The development of the aerodynamic database is detailed in [25]. The primary sources of data were represented by wind tunnel tests, mainly executed in transonic regime, according to its particular interest for the DTFT missions. CFD and simplified engineering methods were used to cross-check wind tunnel data. Simplified methods such as vortex lattice method, panel method and DATCOM were also employed to fill gaps in wind tunnel data and to allow the extension of the database to low subsonic regime. Uncertainty on nominal values was estimated taking into account random experimental errors (repeatability), systematic experimental errors (known and not removable errors) and CFD errors (effect of computational grid, convergence, level of turbulence modeling, boundary conditions).

Maneuverability analysis results, obtained by applying a tool (named PROBAN) implementing the proposed algorithm to the preliminary database in the early phase of the project, made it possible to identify the flight envelope points where aerodynamic database should be refined, because maneuverability requirement was not satisfied on the whole range of uncertainty. Moreover, in such flight envelope points, PROBAN identified which aerodynamic coefficient was characterized by a critical uncertainty, that should be reduced, and which was the amount of uncertainty reduction required to guarantee maneuverability. Following the results provided by PROBAN, further wind tunnel tests and CFD analyses were executed, only focusing on the reduction of critical uncertainties. Thus the use of PROBAN, driving the aerodynamic database refinement process, allowed the maneuverability requirement to be satisfied while dramatically minimize the wind tunnel testing time.

For a fixed flight envelope point, identified by the triple $(\bar{\alpha}, \bar{M}, \overline{Re})$, the formulation of longitudinal rotational maneuverability problem concerns with the solution (with respect to the elevons' position δ_e) of the following constrained nonlinear relation

$$C_m(\bar{\alpha}, \bar{M}, \overline{Re}, \delta_e, \Delta C_m, \Delta C_L, \Delta C_D) = 0 \text{ with } |\delta_e| \leqslant \delta_{e\_m} \tag{15}$$

where C_m is the total pitching moment coefficient evaluated using the centre of gravity (CoG) as reference pole, ΔC_m,ΔC_L and ΔC_D are the aerodynamic uncertainties (related to pitching moment, lift and drag coefficients respectively) and δ_{e_m} is the maximum elevons' deflection deputed to trim the vehicle (for FTB1 it was $\delta_{e\_m} = 15$ deg). For the FTB1 vehicle, the aerodynamic database provided the dependence of C_m on δ_e in tabular nonlinear form, whereas the aerodynamic uncertainties were defined using the nose of the vehicle as a reference pole. For that reason the global pitching moment with respect to the CoG was also dependent on ΔC_L and ΔC_D.

If for each triple $(\Delta C_m, \Delta C_L, \Delta C_D) \in Q$ it exists a $\bar{\delta}_e$ that satisfies Eq. (15), then the vehicle is maneuverable in the examined flight envelope point. Therefore for what concerns this case study, the property to be satisfied by the system is expressed through the inequality $u(q) \leqslant \gamma$, where

$$u(q) = \begin{cases} |\bar{\delta}_e| & \text{if } \exists \bar{\delta}_e : |\bar{\delta}_e| \leqslant \delta_{\text{lim}}, C_m = 0 \\ \delta_{\text{lim}} & \text{otherwise} \end{cases} \tag{16}$$

$q = [\Delta C_D, \Delta C_L, \Delta C_m]$ is the uncertain vector described by the pdf $f(q)$, $\gamma = \delta_{e\_m} = 15$ deg and $\delta_{\text{lim}} = 20$ deg is the elevons' nominal travel range as previously stated.

In this section we report the results of the maneuverability analysis for the flight envelope point identified by the triple $\bar{\alpha} = 15°$, $\bar{M} = 0.96$, $\overline{Re} = 9 \cdot 10^6$. The uncertainty space, provided by the preliminary aerodynamic database, was the hyper-rectangle

$$Q = [-0.055, 0.055] \times [-0.086, 0.086] \times [-0.137, 0.137] \tag{17}$$

This is an input for the maneuverability analysis, being the uncertainty ranges on the global aerodynamic coefficients associated by the aerodynamic database to the examined flight envelope point. The hyper-rectangle found by PROBAN was instead

$$Q_{\text{fin}}^{\text{opt}} = [-0.055, 0.055] \times [-0.086, 0.086] \times [-0.02, 0.137] \tag{18}$$

To evaluate the effectiveness of the results obtained using PROBAN, we have also estimated the allowable aerodynamic region Q_{good} by using the deterministic algorithm PARAN developed by the authors for the same purposes (see [8–10] for details). This algorithm is based on vertices analysis and region bisection techniques. It finds the set where the required property holds by exploring the uncertainty space up to a desired resolution. The latter is related to the maximum number of bisection to carry out with respect to the initial uncertainty space; for example a resolution $\rho = 2^{-3}$ means that the smallest subset of the uncertainty space in which the property is checked has a volume equal to $(1/8)^l$ times the volume of the initial space.

In Fig. 1 the three-dimensional (3D) allowable uncertainty region evaluated by PARAN with $\rho = 2^{-6}$ is shown. The nominal point is (0, 0, 0) and the axis ranges correspond to the preliminary aerodynamic uncertainty ranges. In Fig. 2

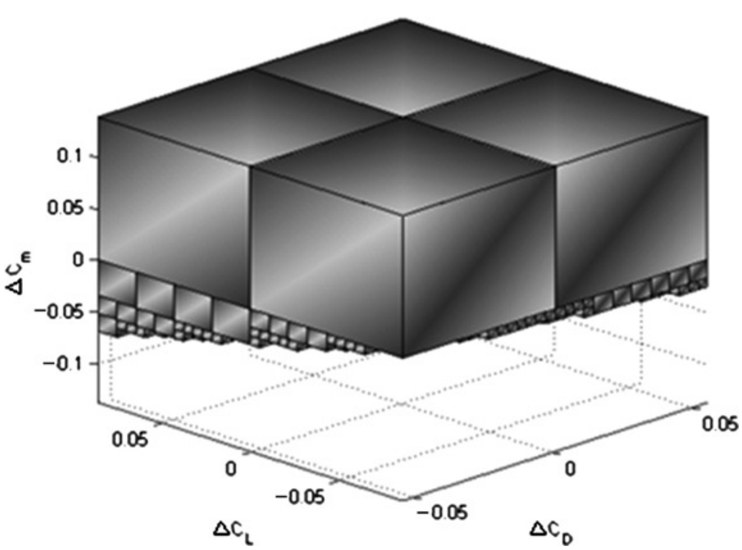

Fig. 1 Set Q_{good} computed by PARAN.

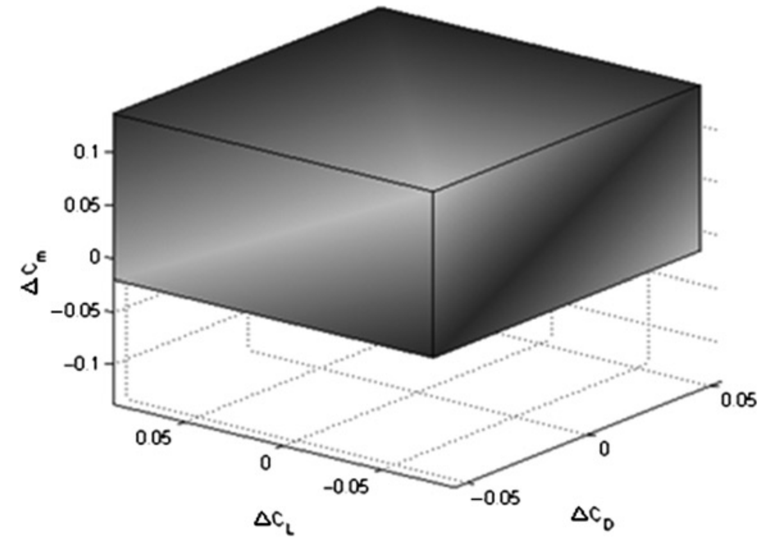

Fig. 2 Hyper-rectangle $Q_{\text{fin}}^{\text{opt}}$ computed by PROBAN.

the allowable set estimated by PROBAN is shown; in particular the figure shows a hyper-rectangle (probabilistically) included in the allowable uncertainties region Q_{good}. It is worth noting that this hyper-rectangle is capable of giving a useful indication about the behavior of the uncertainties, that is, which of the uncertainties are more critical (i.e. require a relevant reduction by means of further wind tunnel tests and CFD analyses). In particular for what concerns FTB1 maneuverability, it is worth to note that the most critical uncertainty was related to the pitching moment coefficient C_{m}.

Because the number of random samples used for the above analysis was $N = 1057$, we can conclude (see Eq. (4)) that the results provided by PROBAN are valid with an accuracy $\varepsilon = 0.005$ and a confidence $\delta = 0.005$, that is, with a probability greater than 0.995 $(1 - \delta)$ the maneuverability is (probabilistically) guaranteed in $Q_{\text{fin}}^{\text{opt}}$, that is

$$\Pr\{u(q) \leqslant \gamma\} \geqslant 1 - \varepsilon, q \in Q_{\text{fin}}^{\text{opt}} \tag{19}$$

with $1 - \varepsilon = 0.995$.

As mentioned previously, a question of great importance is the choice of the pdf to be used to perform the Monte Carlo sampling required by the probabilistic algorithm. Regarding the application discussed in this section a uniform pdf was used according to the considerations made in Sec. II.

Finally it is worth noting that the probabilistic algorithm is not capable of finding the whole admissible uncertainty space Q_{good} because it tries to find the maximum hyper-rectangle included in it. This limitation, however, is counteracted by a significantly better computational efficiency. In fact, the probabilistic algorithm guarantees a noticeable reduction of computational effort. To address the computational complexity of PROBAN, a comparison with PARAN has been carried out with respect to the computation times. Moreover the uncertainties on the CoG position have also been added, in particular z-position Δz_{cg} and x-position Δx_{cg} CoG displacement so to increase the problem size. In particular for $l = 4$ the uncertain vector is $q = [\Delta C_{\text{D}}, \Delta C_{\text{L}}, \Delta C_{\text{m}}, \Delta z_{\text{cg}}]$ and for $l = 5$ $q = [\Delta C_{\text{D}}, \Delta C_{\text{L}}, \Delta C_{\text{m}}, \Delta z_{\text{cg}}, \Delta x_{\text{cg}}]$. In Tables 1 and Table 2 times of execution of PROBAN are reported together with a comparison with those obtained using PARAN. They refer to simulations performed with a Pentium 4, 3.2 GHz computer with 1 Gigabyte RAM. The dimensions of uncertainty space l and the resolution ρ are reported for the deterministic tool while for the probabilistic tool the confidence δ and the accuracy ε are given.

As we can see from the tables, the deterministic tool has a computational burden that quickly increases with the adaptive grid resolution and the number of uncertain parameters. On the contrary, the probabilistic tool PROBAN has a computational burden that does not dramatically increase by improving accuracy and confidence and when the number of uncertain parameters grows. In particular we can note that just for $l = 4$, the deterministic algorithm

Table 1 Computational complexity (probabilistic tool)

l	ε	δ	Time of execution, s
3	0.005	0.005	346
3	0.0025	0.0025	1009
3	0.001	0.001	2081
4	0.005	0.005	480
5	0.005	0.005	686

Table 2 Computational complexity (deterministic tool)

l	ρ	Time of execution, s
3	2^{-4}	220
3	2^{-5}	882
3	2^{-6}	3651
4	2^{-4}	7540
5	2^{-3}	27309

becomes hard to handle from a computational point of view (even with a lower resolution ρ) whereas the probabilistic one allows the computation times to be drastically reduced.

B. Robustness Analysis of FTB 1 FCL to the Initial State Displacement

As stated, DTFT1 mission profile was based on a drop of the FTB1 vehicle from a stratospheric balloon at an altitude between 19 km and 21 km inside a specific target area (safe release zone), lifting off from a launch base located in Arbatax, Sardinia, Italy. Because during the ascent trajectory, the FTB 1 vehicle was subject to a significant rotational motion, the initial angular velocity as well as initial attitude were uncertain. Moreover the drop altitude was not known exactly. As is well known, initial state displacement may heavily influence the overall vehicle trajectory as well as the control system performances such that the mission goals might not be achieved. In view of these considerations, it would be of paramount importance to evaluate the allowable initial state uncertainty still guaranteeing the compliance with system requirements.

As explained before, a robustness analysis has been carried out during the GNC verification phase with the objective of verifying if the control system's performances were achieved in terms of tracking error of angle of attack, sideslip angle, and roll angle despite the initial state displacement in terms of Euler angles, angular velocities and altitude. Therefore for what concerns this case study, the property to be satisfied by the system is expressed through the inequality $u(q) \leqslant \gamma$ where $u(q) = \sqrt{\frac{1}{T}\int_0^T (y(t,q) - y_{\mathrm{ref}}(t))^2\,\mathrm{d}t}$ is the rms (root mean square) tracking error, y is the controlled variable, $q = [\phi_0, \theta_0, \psi_0, p_0, q_0, r_0, h_0]$ are the uncertain parameters, and γ is the maximum acceptable tracking error (defined by mission requirements).

The analysis made it possible easily to identify the subset of initial state uncertainties still guaranteeing the achievement of mission objectives, hence it has been used during DTFT1 mission to decide online whether the vehicle could be safely released depending on its current attitude, angular velocity, and altitude.

Starting from a preliminary analysis of the DTFT1 ascent phase, the following uncertainty range of FTB 1 initial state was selected

$$\begin{aligned} Q = {} & [-180, 180]\mathrm{deg} \times [-90, -82.5]\mathrm{deg} \times [-180, 180]\mathrm{deg} \times [-5, 5]\mathrm{deg}/s \times [-3.5, 3.5]\mathrm{deg}/s \\ & \times [-1.5, 1.5]\mathrm{deg}/s \times [19000, 21000]m \end{aligned} \tag{20}$$

for $\phi_0, \theta_0, \psi_0, p_0, q_0, r_0, h_0$ respectively. The hyper-rectangle found by PROBAN was

$$\begin{aligned} Q_{\mathrm{fin}}^{\mathrm{opt}} = {} & [-120, 120]\mathrm{deg} \times [-90, -86]\mathrm{deg} \times [-180, 155]\mathrm{deg} \times [-3.3, 5]\mathrm{deg}/s \times [-3.5, 3.5]\mathrm{deg}/s \\ & \times [-1.5, 1.5]\mathrm{deg}/s \times [19000, 21000]m \end{aligned} \tag{21}$$

Because the number of random samples used for the above analysis was $N = 1057$, we can conclude (see Eq. (4)) that the results provided by PROBAN are valid with an accuracy $\varepsilon = 0.005$ and a confidence $\delta = 0.005$, that is, with a probability greater than 0.995 $(1 - \delta)$ the success of the mission was (probabilistically) guaranteed in $Q_{\text{fin}}^{\text{opt}}$, that is

$$\Pr\{u(q) \leqslant \gamma\} \geqslant 1 - \varepsilon, q \in Q_{\text{fin}}^{\text{opt}} \tag{22}$$

with $1 - \varepsilon = 0.995$.

Concerning the choice of the pdf used to perform the random sampling, the same considerations of Sec. III.A hold.

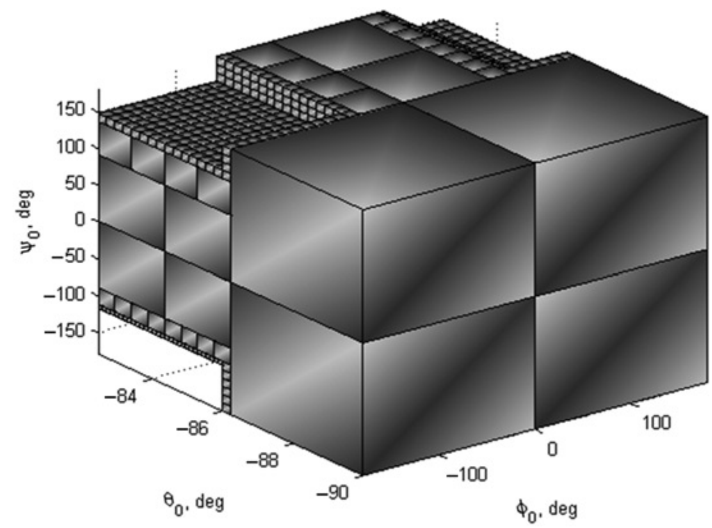

Fig. 3 Set Q_{good} computed by PARAN.

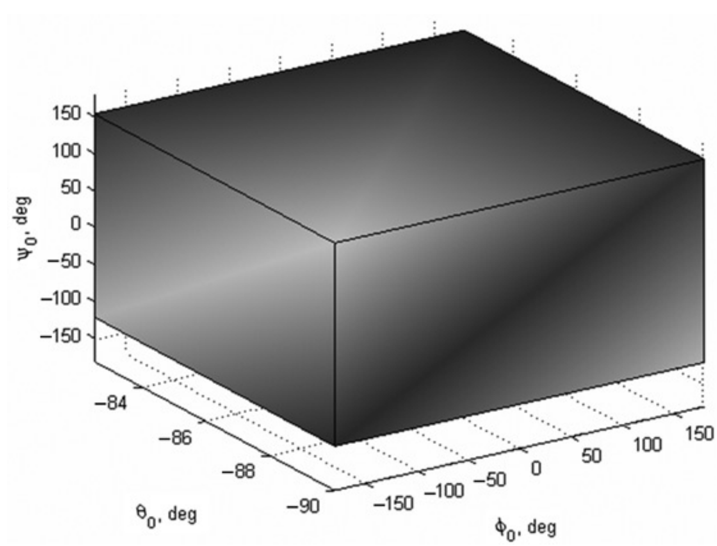

Fig. 4 Hyper-rectangle $Q_{\text{fin}}^{\text{opt}}$ computed by PROBAN.

Table 3 Computational complexity (probabilistic tool)

l	ε	δ	Time of execution, s
3	0.005	0.005	90
4	0.005	0.005	159
5	0.005	0.005	183
6	0.005	0.005	215
7	0.005	0.005	782

Table 4 Computational complexity (deterministic tool)

l	ρ	Time of execution, s
3	2^{-5}	258
4	2^{-5}	17201
5	2^{-3}	7404
6	2^{-2}	2018
7	2^{-2}	15269

Finally to make a comparison with the robustness analysis carried out by the deterministic tool PARAN, we considered a simplified 3D problem. In fact, the results of the deterministic tool cannot be presented in compact form as in Eq. (21) nor they can be plotted for problems with more than 3D uncertainties. Therefore we considered the uncertainties on the vehicle attitude, letting the remaining variables assume their nominal value, thus obtaining the following initial uncertainty space

$$Q = [-180, 180]\text{deg} \times [-90, -82.5]\text{deg} \times [-180, 180]\text{deg} \tag{23}$$

In Fig. 3 and Fig. 4, the subsets found by PARAN with resolution $\rho = 2^{-6}$ and PROBAN with $N = 1057$ are depicted respectively. It is worth noting that even if the probabilistic algorithm is not capable of finding the whole admissible uncertainty space, it guarantees a significantly better computational efficiency. In this regard, in Table 3 and Table 4, the times of execution of PROBAN and PARAN are reported respectively for different dimensions of the uncertainty space. In particular for $l = 3$ the uncertain vector is $q = [\phi_0, \theta_0, \psi_0]$, for $l = 4$ $q = [\phi_0, \theta_0, \psi_0, p_0]$, for $l = 5$ $q = [\phi_0, \theta_0, \psi_0, p_0, q_0]$, etc. The results refer to simulations performed with a Pentium 4 3.2 GHz computer with 1 Gigabyte RAM.

IV. Conclusion

A new algorithm for the analysis of an uncertain system in a probabilistic framework has been proposed. It identifies a hyper-rectangle included in the uncertainty parameter space in which a given property is satisfied with pre-assigned accuracy and confidence. The identified hyper-rectangle is suboptimal in the sense that the procedure finds a local maximum volume hyper-rectangle included in the actual uncertainty region where the property is satisfied.

The proposed algorithm overcomes the main limitations of the common approaches to the robustness analysis problem, namely the computational complexity, the inapplicability to every kind of system, and the conservatism of the solution.

In the paper two practical applications have been described. The first is the identification (in the early design phase of the project) of the flight envelope points where the preliminary aerodynamic database should be refined (and the amount of uncertainty reduction required) by means of further wind tunnel tests and CFD analyses, because maneuverability was not guaranteed on the whole preliminary range of uncertainty. The second is the estimation of FTB1 drop initial state displacement for a safe flight execution. Moreover a comparison with the results obtained by a similar tool based on a deterministic approach to the robustness analysis problem has been made. Although a small part of the admissible uncertainty space was lost when the proposed algorithm was used (with respect to the result of the deterministic tool), this drawback is widely compensated for by the reduction of the computational burden that is polynomial increasing with the dimension of the uncertainty space. Moreover, the presented results clearly demonstrate that the proposed algorithm can deal with every type of systems and property to be verified as well as with every uncertainty structure.

Comparing the above features with similar tools and current industrial practice, we can conclude that the algorithm is very effective in every application where it is necessary to verify or, more generally, to analyze the performance of a highly uncertain system.

References

[1] Fielding, C., Varga, A., Bennani, S., and Selier, M. (eds), *Advanced Techniques for Clearance of Flight Control Laws*, Springer Verlag, Berlin, 2002, Chaps. 1–7.

[2] Motoda, T., and Miyazawa, Y., "Identification of Influential Uncertainties in Monte Carlo Analysis," *Journal of Spacecraft and Rockets*, Vol. 39, No. 4, 2002, pp. 615–623.
doi: 10.2514/2.3851

[3] Skogestad, S., and Postlethwaite, I., *Multivariable Feedback Control: Analysis and Design*, Wiley, New York, NY, 1996.

[4] Packard, A., and Doyle, J. C., "The Complex Structured Singular Value," *Automatica* Vol. 29, No. 1, 1993, pp. 71–109.
doi: 10.1016/0005-1098(93)90175-S

[5] Zhou, K., Doyle, J. C., and Glover, K., *Robust and Optimal Control*, Prentice Hall, Upper Saddle River, NJ, 1996.

[6] Bhattacharyya, S. P., Chapellat, H., and Keel, L.H., *Robust Control: The Parametric Approach*, Prentice Hall, Upper Saddle River, NJ, 1995.

[7] Bartlett, A. C., Hollot, C. V., and Huang Lin, "Root Locations of an Entire Polytope of Polynomials: It Suffices to Check the Edges," *Mathematics of Control, Signals, and Systems*, Vol. 1, No. 1, Feb. 1988, pp. 61–71.
doi: 10.1007/BF02551236

[8] Vitale, A., Tancredi, U., Verde, L., Filippone, E., and Corraro, F., "A Tool For Space Vehicle Uncertainty Ranges Estimation," *55th International Astronautical Congress (IAC)*, Oct. 2004, Curran Associates Inc., IAC-04-U.3.B.05, Dec. 2006.

[9] Verde, L., and Corraro, F., "A Polynomial-Based Clearance Method," *Advanced Techniques for Clearance of Flight Control Laws*, Springer Verlag, Berlin, 2002, pp. 77–88.
doi: 10.1007/3-540-45864-6_5

[10] Verde, L., and Corraro, F., "Polynomial-Based Clearance of Eigenvalue Criteria," *Advanced Techniques for Clearance of Flight Control Laws*, Springer Verlag, Berlin, 2002, pp. 333–353.
doi: 10.1007/3-540-45864-6_19

[11] Blondel, V. D., and Tsitsiklis, J. N., "A Survey of Computational Complexity results in Systems and Control," *Automatica* Vol. 36, No. 9, 2000, pp. 1249–1274.
doi: 10.1016/S0005-1098(00)00050-9

[12] Stengel, R. F., "Some Effects of Parameter Variations on the Lateral-Directional Stability of Aircraft," *Journal of Guidance and Control*, Vol. 3, No. 2, 1980, pp. 124–131.
doi: 10.2514/3.55959

[13] Ray, L. R., and Stengel, R. F., "Application of Stochastic Robustness to Aircraft Control Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 6, 1991, pp. 1251–1259.

[14] Vidyasagar, M., "Statistical Learning Theory and Randomized Algorithms for Control," *IEEE Control System Magazine*, Vol. 18, No. 6, Dec. 1998, pp. 69–85.
doi: 10.1109/37.736014

[15] Calafiore, G. C., Dabbene, F., and Tempo, R., "Randomized Algorithms for Probabilistic Robustness with Real and Complex Structured Uncertainty," *IEEE Transactions on Automatic Control*, Vol. 45, No. 12, Dec. 2000, pp. 2218–2235.
doi: 10.1109/9.895560

[16] Khargonekar, P., and Tikku, A., "Randomized algorithms for Robust Control Analysis and Synthesis have Polynomial Complexity," *Proceedings of the 35th Conference on Decision and Control*, IEEE, New York, NY, Vol. 3, 11–13 Dec. 1996, pp. 3470–3475. doi: 10.1109/CDC.1996.573700

[17] Tempo, R., Calafiore, G. C., and Dabbene, F., *Randomized Algorithms for Analysis and Control of Uncertain Systems*, Springer Verlag, New York, NY, 2005, Chaps. 8, 11, 12, 13

[18] Chen, X., and Zhou, K., and Aravena, J. L., "Fast Universal Algorithms for Robustness Analysis," *Proceedings of the 42nd Conference on Decision and Control*, IEEE, New York, NY, Vol. 2, 9–12 Dec. 2003, pp. 1926–1931. doi: 10.1109/CDC.2003.1272897

[19] Barmish, B. R., and Polyak, B. T., "The Volumetric Singular Value and Robustness of Feedback Control Systems," *Proceedings of the 32nd Conference on Decision and Control*, IEEE, New York, NY, Vol. 1, 15–17 Dec. 1993, pp. 521–522. doi: 10.1109/CDC.1993.325093

[20] Tempo, R., Bai, E. W., and Dabbene, F., “Probabilistic Robustness Analysis: Explicit Bounds for the Minimum Number of Samples,” *Systems & Control Letters*, Vol. 30, 1997, pp. 237–242. doi: 10.1016/S0167-6911(97)00005-4
[21] Lagoa, C. M., and Barmish, B. R., “Distributionally Robust Monte Carlo Simulation: A Tutorial Survey,” *Proceedings of the IFAC World Congress*, Elsevier, Amsterdam, Vol. 15, Part 1, July 2002.
[22] Barmish, B. R., and Lagoa, C. M., “The Uniform Distribution: A Rigorous Justification for Its Use in Robustness Analysis,” *Mathematics of Control, Signals, and Systems*, Vol. 10, No. 3, 1997, pp. 203–222. doi: 10.1007/BF01211503
[23] Bai, E. W., Tempo, R., and Fu, M., “Worst-Case Properties of the Uniform Distribution and Randomized Algorithms for Robustness Analysis,” *Mathematics of Control, Signals, and Systems (MCSS)*, Vol. 11, No. 3, Sep. 1998, pp. 183–196. doi: 10.1007/BF02741890
[24] Zhou, T., and Feng, C., “On Control System Design Using Random Samples of Contractive Block Toeplitz Matrices,” *International Journal of Control*, Vol. 80, No. 4, 2007, pp. 624–635. doi: 10.1080/00207170601129036
[25] Rufolo, G. C., Roncioni, P., Marini, M., Votta, R., and Palazzo, S., “Experimental and Numerical Aerodynamic Data Integration and Aerodatabase Development for the PRORA-USV-FTB_1 Reusable Vehicle,” *4th AIAA/AHI Space Plane and Hypersonic Systems and Technologies Conference*, AIAA, Reston, VA, 2006, AIAA-2006-8031.

Christopher Rouff
Associate Editor